

International Journal of Modern Physics A
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ANALYTICAL EXPRESSIONS OF 3 AND 4-LOOP SUNRISE FEYNMAN INTEGRALS AND 4-DIMENSIONAL LATTICE INTEGRALS

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Received Day Month Year

Revised Day Month Year

In this paper we continue the work begun in 2002 on the identification of the analytical expressions of Feynman integrals which require the evaluation of multiple elliptic integrals. We rewrite and simplify the analytical expression of the 3-loop self-mass integral with three equal masses and on-shell external momentum. We collect and analyze a number of results on double and triple elliptic integrals. By using very high-precision numerical fits, for the first time we are able to identify a very compact analytical expression for the 4-loop on-shell self-mass integral with 4 equal masses, that is one of the master integrals of the 4-loop electron $g-2$. Moreover, we fit the analytical expressions of some integrals which appear in lattice perturbation theory, and in particular the 4-dimensional generalized Watson integral.

Keywords: Feynman diagram; master integral; elliptic integral; lattice green function; Watson integral.

PACS numbers: 02.60.Jh, 12.20.Ds, 12.38.Gc

1. Introduction

Analytical expressions of many Feynman diagrams contain polylogarithmic functions of various kinds (Nielsen polylogarithms, Harmonic polylogarithms¹, Harmonic sums^{2,3}, multiple zeta values, Euler sums⁴, etc...). But there exist Feynman integrals which cannot be described only in terms of polylogarithms.

At two-loop level, the discontinuity of the off-shell massive “sunrise” diagram with different masses is expressed by elliptic functions. At three (or more) loop level the situation worsens. These diagrams contain nested multiple elliptic integrals; the current mathematical knowledge of such integrals is scarce or missing. At this preliminary stage, “*experimental mathematics*” is the best tool. In other words, high-precision numerical values of the integrals are fitted⁵ with various candidate analytical expressions until an agreement is found. The equalities are then checked up to hundreds or thousands of digits. In this way the right analytical expression is identified beyond any reasonable doubt. This may be the starting point of the subsequent search for a rigorous proof of the result, task which may take months of hard work^{6,7,8}.

In Ref. 5 we work out a very high-precision value of the 3-loop scalar master integral of the “sunrise” diagram S_3 of Fig.1; then we fit that value with products of elliptic integrals, checking the equality with a precision of thousand of digits.

In this paper we continue the work on the 3-loop integral, and we simplify the 3-loop analytic result by using some identities between elliptic integrals. Next we review and develop the approach used for fitting and identifying the 3-loop integral and we apply it to the 4-loop scalar master integral of the “sunrise” diagram S_4 of Fig.1. The analytical calculation of this master integral is also of physical interest, because it is the simplest non-trivial master integral of the 4-loop electron g -2 (of which a high-precision numerical calculation is under way⁹). Very likely, the analytical constants which appear in the expression of S_4 should also appear in the analytical expression of 4-loop g -2. We calculate an high-precision value of S_4 and we are able to fit this value with an expression containing two new elliptic constants, checking the equality with a precision of thousand of digits.

We apply this procedure also to the values of some 4-dimensional lattice integrals, and we identify their analytical expressions; surprisingly, they contain the same elliptic constants of the 4-loop integral.

The plan of the paper is the following: In section 2 we simplify the results of Ref. 5 by using identities between elliptic integrals. In section 3 we study the 4-loop “sunrise” integral. We collect a number of results on a “simplified” version of the integrals involved. Then we use these results as a guide for identifying the candidate analytical expressions suitable for fitting the 4-loop results. In section 4 we show the analytical results found for the 4-loop integrals. In section 5 we fit the values of some 4-dimensional lattice integrals with the same analytical constants discovered in the 4-loop integrals. In section 6 we give our conclusions.

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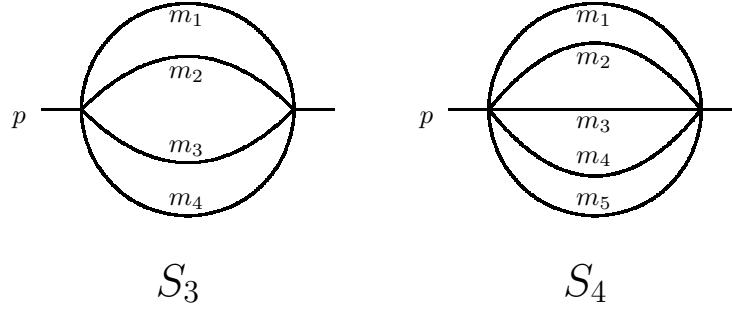


Fig. 1. Three-loop and four-loop self-mass diagrams.

2. Three-loop single-scale self-mass integral

2.1. The results of Ref. 5

In Ref. 5 we considered the Feynman diagram $S_3(p^2, m_1^2, m_2^2, m_3^2, m_4^2, D)$ with equal masses $m_j = 1$, and on-shell external momentum (see Fig.1)

$$S_3(-1, 1, 1, 1, 1, D) = \int \frac{[d^D q_1] [d^D q_2] [d^D q_3]}{(q_1^2 + 1)(q_2^2 + 1)(q_3^2 + 1)((p - q_1 - q_2 - q_3)^2 + 1)} , \quad p^2 = -1 , \quad (1)$$

where

$$[d^D q] = \frac{d^D q}{\pi^{D/2} \Gamma\left(3 - \frac{D}{2}\right)} . \quad (2)$$

By using an hyperspherical representation for the integral, we found that the value of S_3 could be expressed as a sum of various double elliptic integrals, the simplest being

$$A_3 = \int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, l, -1)R(m, -1, -1)} = 2.641\,379\,476\,074\,689\,431\,349\dots , \quad (3)$$

$$R(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz} . \quad (4)$$

We were not able to calculate A_3 in analytical form. Therefore we evaluated it at very high precision and we tried to fit the numerical value with various kinds of analytical expressions. In Ref. 5 we found that

$$A_3 = K(w_-)K(w_+) , \quad w_\pm = \frac{z_\pm}{z_\pm - 1} , \quad z_\pm = -(2 - \sqrt{3})^4(4 \pm \sqrt{15})^2 , \quad (5)$$

where K is the first of the two elliptic integrals

$$K(m) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-mt^2}}, \quad E(m) = \int_0^1 \frac{dt \sqrt{1-mt^2}}{\sqrt{1-t^2}}. \quad (6)$$

We were able to fit the values of S_3 in 2 and 4 dimensions:^a

$$S_3(-1, 1, 1, 1, 1, D=2) = \frac{4\pi}{\sqrt{15}} K(w_-) K(w_+), \quad (7)$$

$$\begin{aligned} S_3(-1, 1, 1, 1, 1, D=4-2\epsilon) &= 2\epsilon^{-3} + \frac{22}{3}\epsilon^{-2} + \frac{577}{36}\epsilon^{-1} + \frac{4\pi}{\sqrt{15}} \left(\frac{35}{8}\pi \right. \\ &+ \frac{131}{12} K(w_-) K(w_+) - \frac{7}{2} (E(1-w_-)E(1-w_+) + 5E(w_-)E(w_+)) \Big) + \frac{6191}{216} + O(\epsilon). \end{aligned} \quad (8)$$

Eq.(5), Eq.(7) and Eq.(8) were checked up to 30000, 40000 and 1200 digits, respectively.

2.2. New relations between elliptic integrals

Now we note that the arguments w_{\pm} of the elliptic integrals are *singular values*. In the context of elliptic integrals k_r is called singular value if

$$\frac{K(1-k_r)}{K(k_r)} = \sqrt{r}, \quad (9)$$

where r is an integer or a rational number. The arguments w_{\pm} of elliptic integrals are singular values for $r=15$ and $r=5/3$ respectively,

$$w_- = k_{15}, \quad w_+ = k_{5/3},$$

that is

$$\frac{K(1-k_{15})}{K(k_{15})} = \sqrt{15}, \quad \frac{K(1-k_{5/3})}{K(k_{5/3})} = \sqrt{\frac{5}{3}}, \quad (10)$$

$$\frac{K(k_{5/3})}{K(k_{15})} = \frac{\sqrt{15} - \sqrt{3}}{2}. \quad (11)$$

The values of elliptic integrals of second kind of Eq.(8) are obtained following Ref. 10

$$E(k_r) = \frac{\pi}{4\sqrt{r}K(k_r)} + K(k_r) \left(1 - \frac{\alpha_r}{\sqrt{r}} \right), \quad (12)$$

$$E(1-k_r) = \frac{\pi}{4K(k_r)} + K(k_r)\alpha_r. \quad (13)$$

$$(14)$$

^a The two-dimensional value (7) was calculated together with (8), but not published in Ref. 5.

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By using the values¹⁰

$$\alpha_{15} = \frac{\sqrt{15} - \sqrt{5} - 1}{2}, \quad \alpha_{5/3} = \frac{\sqrt{15} - \sqrt{5} + 1}{6}, \quad (15)$$

$$K(k_{15}) = \sqrt{\frac{(\sqrt{5} + 1)P}{240\pi}}, \quad (16)$$

where

$$P \equiv \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right), \quad (17)$$

and expressing $K(k_{5/3})$ by using Eq.(11), we are able to rewrite Eq.(7) and Eq.(8) in the very compact form

$$S_3(-1, 1, 1, 1, 1, D = 2) = \frac{P}{40\sqrt{3}\pi}, \quad (18)$$

$$S_3(-1, 1, 1, 1, 1, D = 4 - 2\epsilon) = 2\epsilon^{-3} + \frac{22}{3}\epsilon^{-2} + \frac{577}{36}\epsilon^{-1} + \frac{6191}{216} - \frac{14\sqrt{5}\pi^4}{P} - \frac{\sqrt{5}}{900}P + O(\epsilon). \quad (19)$$

Eqs.(10)-(15) were also shown by David Broadhurst in his beautiful talk given in Bielefeld¹¹. After the talk, our unpublished results (17)-(19) were shown to David Broadhurst.

2.3. The path to Eq.(5)

For sake of completeness we recall here some unpublished observations that suggested us the form of Eq.(5). First we calculated analytically the *simplest* double elliptic integral:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}\sqrt{1-x^2y^2}} = \left[K\left(\frac{1}{2}\right) \right]^2 = 3.437\,592\,909\dots, \quad (20)$$

which factorizes in the square of the elliptic integral K .

Then we observed that in the diagram $S_3(-1, 1, 1, 1, 1)$ the value of the square of the external momentum p^2 is -1 , which is different from the threshold ($p^2 = -16$) or the pseudothresholds ($p^2 = -4, 0$). We expected that the analytical expression is much simpler for the on-threshold diagram than for the off-threshold diagram. So we considered the above graph with one mass changed: $S_3(-1, 1, 1, 1, 2)$. Now the value of $p^2 = -1$ is on a pseudothreshold (which are at $p^2 = -1, -9, -25$). The integral analogous to A_3 is

$$A'_3 = \int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, l, -4)R(m, -1, -1)} = 1.474\,585\,992\dots \quad (21)$$

We were able to fit the numerical value of A'_3 with

$$A'_3 = \frac{1}{\sqrt{3}} \left(\frac{\pi}{2}\right)^2 {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{4}\right) = \frac{1}{\sqrt{3}} \left[K\left(\frac{2-\sqrt{3}}{4}\right) \right]^2 = \frac{\Gamma^6\left(\frac{1}{3}\right)}{2^{\frac{14}{3}}\pi^2}. \quad (22)$$

Subsequently, as we expected the form of A_3 to be more complicated than A'_3 , we tried also products of K with different arguments, and we found Eq.(5).

3. Four-loop single-scale self-mass integral

Now we consider the single-scale 4-loop self-mass diagram $S_4(p^2, m_1^2, m_2^2, m_3^2, m_4^2, D)$ of Fig.1, in the case of equal masses $m_j = 1$ and on shell external momentum $p^2 = -1$. This diagram has 3 master integrals, the simplest being

$$S_4(-1, 1, 1, 1, 1, D) = \int \frac{[d^D q_1] [d^D q_2] [d^D q_3] [d^D q_4]}{(q_1^2 + 1)(q_2^2 + 1)(q_3^2 + 1)(q_4^2 + 1)((p - q_1 - q_2 - q_3 - q_4)^2 + 1)} . \quad (23)$$

As already said in the introduction, this is also one of the several master integrals which appear in the calculation of 4-loop g -2.

The first observation is that, fortunately, the value of $p^2 = -1$ in S_4 is already on a pseudothreshold (which are $p^2 = -1, -9, -25$); therefore to simplify the analytical structure of the integral we have not to use the mass change of the previous section 2.3.

3.1. High-precision numerical values

We need an high-precision numerical value of this integral in order to obtain a meaningful fit. This is worked out by using the difference equation method presented in Refs. 13, 14. Summarizing, one raises to n one denominator of Eq.(23)

$$X_4(n) = \int \frac{[d^D q_1] [d^D q_2] [d^D q_3] [d^D q_4]}{(q_1^2 + 1)^n (q_2^2 + 1)(q_3^2 + 1)(q_4^2 + 1)((p - q_1 - q_2 - q_3 - q_4)^2 + 1)} . \quad (24)$$

The function $X_4(n)$ satisfies the fourth-order difference equation

$$p_1 X_4(n+3) + p_2 X_4(n+2) + p_3 X_4(n+1) + p_4 X_4(n) + p_5 X_4(n-1) = 24(D-2)^4 J^3(1) J(n) , \quad (25)$$

where

$$\begin{aligned} p_1 &= -768n(n+1)(n+2)(n-2D+5) , \\ p_2 &= 128n(n+1)(11n^2 + (63-26D)n + 11D^2 - 57D + 76) , \\ p_3 &= 4n(-129n^3 + (294D-588)n^2 + (-148D^2 + 592D - 567)n \\ &\quad + 8D^3 - 48D^2 + 46D + 36) , \\ p_4 &= 2(-60n^4 + (294D-468)n^3 + (-485D^2 + 1499D - 1164)n^2 + (308D^3 \\ &\quad - 1363D^2 + 2015D - 996)n - 60D^4 + 326D^3 - 658D^2 + 584D - 192) , \\ p_5 &= -(n-D+1)(n-2D+3)(2n-3D+4)(2n-5D+8) , \end{aligned} \quad (26)$$

and J is the one-loop integral

$$J(n) = \int \frac{[d^D q_1]}{(q_1^2 + 1)^n} . \quad (27)$$

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Eq.(25) contains in the r.h.s. the integral obtained from S_4 by contracting one line, which factorizes into the product of 4 one-loop tadpoles. The solution of Eq.(25) compatible with the large- n boundary condition $X_4(n) \propto n^{-D/2}$ can be written as $X_4(n) = C_1 X_1^{HO}(n) + C_2 X_2^{HO}(n) + X^{NH}(n)$. The functions X_1^{HO} , X_2^{HO} and X^{NH} are respectively the two solutions of the homogeneous equation compatible with the above large- n behaviour and one particular solution of the nonhomogeneous equation Eq.(25). The constants C_1 and C_2 are obtained from the 3-loop self-mass integrals belonging to the diagram obtained from S_4 by deleting one line, that is S_3 . The amount of calculations needed to work out and solve the systems of difference equations is high, so that the calculations have been performed by means of an automatic tool, the program **SYS** described in Ref. 13.

In two dimension one finds

$$S_4(-1, 1, 1, 1, 1, 1, D = 2) = 40.2451219019305821264798187417 \dots \quad (28)$$

and in the limit $D \rightarrow 4$

$$S_4(-1, 1, 1, 1, 1, 1, D = 4 - 2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} \quad (29)$$

$$\begin{aligned} & - 141.72215618664768694996791 - 521.14654568600250441775466\epsilon \\ & - 3347.9933650782886117865341\epsilon^2 - 17951.3774774809944931097622\epsilon^3 \\ & - 101753.8165331173182139560386\epsilon^4 + O(\epsilon^5) . \end{aligned} \quad (30)$$

All the above numerical constants were calculated with a precision of over 2400 digits; for the sake of space we show here only the first 25.

3.2. Triple elliptic integrals

By using an hyperspherical representation for the integral, $S_4(-1, 1, 1, 1, 1, 1, D = 2)$ and the finite part of $S_4(-1, 1, 1, 1, 1, 1, D = 4 - 2\epsilon)$ contain *triple* elliptic integrals, the simplest being

$$A_4 = \int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, l, -1)} \int_0^\infty \frac{dr}{R(m, r, -1)R(r, -1, -1)} = 8.749\,361\,490\dots \quad (31)$$

We prefer to redistribute the arguments of the R functions and consider the similar integral

$$\int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, -1, -1)} \int_0^{(\sqrt{l}-\sqrt{m})^2} \frac{dr}{R(r, l, m)R(r, -1, -1)} = i8.749\,361\,490\dots = iA_4 , \quad (32)$$

and the companion integral

$$B_4 = \int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, -1, -1)} \int_{(\sqrt{l}-\sqrt{m})^2}^{(\sqrt{l}+\sqrt{m})^2} \frac{dr}{R(r, l, m)R(r, -1, -1)} = 9.607\,815\,129\dots, \quad (33)$$

where $r_\pm = (\sqrt{l} \pm \sqrt{m})^2$ are the two zeroes of $R(r, l, m)$. A_4 and B_4 are the 4-loop integrals analogues of the 3-loop integral A_3 , and we have to find their analytical expressions.

3.3. Simplifying the problem

First of all we consider the simplest triple elliptic integral with structure similar to Eq.(32):

$$A = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}} \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-x^2y^2z^2}} = 4.335\,593\,665\dots \quad (34)$$

By changing one limit of integration over z to the zero of $1 - x^2y^2z^2$ we obtain a companion integral analogous to Eq.(33)

$$B = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}} \int_1^{1/xy} \frac{dz}{\sqrt{z^2-1}\sqrt{1-x^2y^2z^2}} = 6.997\,563\,016\dots \quad (35)$$

We expect that the study of the simpler constants A and B can help us understand the analytical expressions of A_4 and B_4 . Integrating over z

$$A = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}} K(x^2y^2), \quad (36)$$

$$B = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}} K(1-x^2y^2). \quad (37)$$

Integrating over y and x

$$A = \int_0^1 \frac{dx}{\sqrt{1-x^2}} K^2\left(\frac{1-\sqrt{1-x^2}}{2}\right) = \left(\frac{\pi}{2}\right)^3 {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1\right), \quad (38)$$

$$B = \int_0^1 \frac{dx}{\sqrt{1-x^2}} K\left(\frac{1-\sqrt{1-x^2}}{2}\right) K\left(\frac{1+\sqrt{1-x^2}}{2}\right). \quad (39)$$

No analytical expression is known for ${}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1\right)$. Trying to understand the reason of that, we study the following family of integrals:

$$\int_0^1 dt K^m(t) K^n(1-t) \left(\frac{1}{\sqrt{t}}\right)^\alpha \left(\frac{1}{\sqrt{1-t}}\right)^\beta. \quad (40)$$

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We consider here only the integrals which have results containing elliptic constants. At the value $m + n = 1$ there is the integral

$$\int_0^1 dt \frac{K(t)}{\sqrt{t(1-t)}} = 2K^2\left(\frac{1}{2}\right), \quad (41)$$

equivalent to Eq.(20); note that it factorizes in a square of $K(1/2) = \Gamma^2(1/4)/(4\sqrt{\pi})$.

At the value $m + n = 2$ we find numerically that the six integrals

$$\int_0^1 dt \frac{K^2(t)}{\sqrt{t}} = B, \quad (42)$$

$$\int_0^1 dt \frac{K^2(t)}{\sqrt{1-t}} = 2B, \quad (43)$$

$$\int_0^1 dt \frac{K^2(t)}{\sqrt{t(1-t)}} = 4A, \quad (44)$$

$$\int_0^1 dt \frac{K(t)K(1-t)}{\sqrt{t}} = \int_0^1 dt \frac{K(t)K(1-t)}{\sqrt{1-t}} = 2A, \quad (45)$$

$$\int_0^1 dt \frac{K(t)K(1-t)}{\sqrt{t(1-t)}} = 2B, \quad (46)$$

can be expressed in terms of A and B . At the value $m + n = 3$ we find a surprise: six integrals factorizes into the fourth power of $K(1/2)$.

$$\int_0^1 dt K^3(t) = \frac{4}{5}K^4\left(\frac{1}{2}\right), \quad (47)$$

$$\int_0^1 dt \frac{K^3(t)}{\sqrt{t}} = \frac{6}{5}K^4\left(\frac{1}{2}\right), \quad (48)$$

$$\int_0^1 dt \frac{K^3(t)}{\sqrt{1-t}} = 4K^4\left(\frac{1}{2}\right), \quad (49)$$

$$\int_0^1 dt K^2(t)K(1-t) = \frac{2}{3}K^4\left(\frac{1}{2}\right), \quad (50)$$

$$\int_0^1 dt \frac{K^2(t)K(1-t)}{\sqrt{t}} = \frac{4}{3}K^4\left(\frac{1}{2}\right), \quad (51)$$

$$\int_0^1 dt \frac{K^2(t)K(1-t)}{\sqrt{1-t}} = 2K^4\left(\frac{1}{2}\right). \quad (52)$$

The analytical results (42)-(52) have been fitted numerically and checked up to 200 digits of precision. We find results factorized at level 2 and 4, but not at level 3 (odd).

This behaviour reminds us of the non-factorization of values of Riemann ζ -function at odd integers, and suggest to consider the constants A and B as irreducible objects. We can relate A and B to multidimensional ζ -like quadruple series. Let us consider the integrals

$$I_m = \int_0^1 dt \frac{K^2(t)}{\sqrt{1-t}} \left(\frac{K(1-t)}{K(t)} \right)^m. \quad (53)$$

These integrals correspond to the integrals (43), (45), (42), for $m = 0, 1, 2$, respectively, and their values are $I_0 = 2B$, $I_1 = 2A$, $I_2 = B$. We apply the change of variable

$$q = \exp(-\pi K(1-t)/K(t)) \quad \text{or equivalently} \quad 1-t = (\theta_4(q)/\theta_3(q))^4, \quad (54)$$

where $\theta_j(q)$ are the Jacobi Theta Functions, then

$$I_m = \pi^{2-m} \int_0^1 dq \left(\theta_4^2(q) \theta_3(q) \frac{d}{dq} \theta_3(q) - \theta_3^2(q) \theta_4(q) \frac{d}{dq} \theta_4(q) \right) (-\log q)^m. \quad (55)$$

Expanding in series the θ functions, and integrating over q term-by-term, I_m becomes a quadruple series

$$I_m = m! \pi^{2-m} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} ', \frac{(-1)^{i+j}(k^2 - i^2)}{(i^2 + j^2 + k^2 + l^2)^{m+1}}; \quad (56)$$

for $m = 2$ the series converges, so that

$$B = 2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} ', \frac{(-1)^{i+j}(k^2 - i^2)}{(i^2 + j^2 + k^2 + l^2)^3}, \quad (57)$$

where the prime means that the origin $i = j = k = l = 0$ must be excluded in the summation.

3.4. Integrals of products of homogeneous solutions

Coming back to the 4-loop integral, if we close the 4-loop self-mass diagram S_4 by connecting together the two external lines, we obtain a 5-loop vacuum diagram. The 5-loop vacuum diagram can be decomposed into two 2-loop self-mass diagrams connected together. Therefore in $D = 2$ dimensions its value is given by the integral of the square of the 2-loop self-mass diagram $\int d^2p S_2^2(p^2, 1, 1, 1)$. The vacuum diagram is expected to have the same analytical structure of S_4 , but with higher transcendentality. $S_2(u)$ satisfies a second order nonhomogeneous differential equation (see Ref. 12 for more details). The corresponding homogeneous differential equation has two solutions $J_1(u)$ and $J_2(u)$. In order to reduce the transcendentality we may substitute $S_2(u)$ with $J_1(u)$ or $J_2(u)$. The analytical expressions of J_1 and J_2 are

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given in the Appendix of Ref. 12; for example, if $0 \leq u \leq 1$ they read

$$\begin{aligned} J_1(u) &= \frac{1}{\sqrt{(1+\sqrt{u})^3(3-\sqrt{u})}} K(a(u)) , \\ J_2(u) &= \frac{1}{\sqrt{(1+\sqrt{u})^3(3-\sqrt{u})}} K(1-a(u)) , \\ a(u) &= \frac{(1-\sqrt{u})^3(3+\sqrt{u})}{(1+\sqrt{u})^3(3-\sqrt{u})} . \end{aligned} \quad (58)$$

In Ref. 12 we found also that

$$\int_0^1 du J_1(u) = \text{Cl}_2(\pi/3) . \quad (59)$$

We consider here the integrals of products of J_i : we have checked up to 2400-digits of precision the following equalities:

$$\int_0^1 du J_1^2(u) = \frac{1}{8} A_4 , \quad (60)$$

$$\int_0^1 du J_2^2(u) = \frac{3}{4} A_4 , \quad (61)$$

$$\int_0^1 du J_1(u) J_2(u) = \frac{1}{4} B_4 , \quad (62)$$

and

$$\int_1^9 du J_1^2(u) = \frac{3}{8} A_4 , \quad (63)$$

$$\int_1^9 du J_2^2(u) = \frac{9}{8} A_4 , \quad (64)$$

$$\int_1^9 du J_1(u) J_2(u) = \frac{1}{2} B_4 . \quad (65)$$

Therefore we have identified some one-dimensional integral representations of the numerical constants A_4 and B_4 .

3.5. The key observation

Eqs.(60)-(65) are not satisfactory elementary definitions of A_4 and B_4 , because of the complexity of the arguments of the K function in Eq.(58). Our aim is to find out integral representations of A_4 and B_4 as simple as Eq.(42) and Eq.(45). We make a comparison between the integrals of products of K of the family (40) and the integrals of products of J_i . Eq.(59) corresponds to

$$\int_0^1 dt \frac{K(t)}{\sqrt{1-t}} = \frac{\pi^2}{2} \quad (66)$$

We try to modify the integrand of (66) so that the result contains the constant $\text{Cl}_2(\pi/3)$. Our many year experience with the analytical calculation of 3-loop g -2

suggests that such a constant is usually associated with integrals containing the polynomial $1+t+t^2$ in the denominator. This is a factor of $1-t^3$. Therefore we try to consider a “cubic” modification of the usual elliptic integral $K(m)$. One fruitful choice is

$$K_c(m) = \int_0^1 \frac{dt}{\sqrt[3]{(1-t^3)(1-mt^3)^2}}, \quad E_c(m) = \int_0^1 \frac{dt \sqrt[3]{1-mt^3}}{\sqrt[3]{1-t^3}}, \quad (67)$$

or, equivalently, expressing them in terms of the hypergeometric function

$$K_c(m) = \frac{2\pi}{\sqrt{27}} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; m\right), \quad E_c(m) = \frac{2\pi}{\sqrt{27}} {}_2F_1\left(\frac{1}{3}, -\frac{1}{3}; m\right). \quad (68)$$

Now we calculate the numerical values of the integrals obtained by replacing K with K_c in Eq.(40), and we look for relations with the constants A_4 and B_4 . Luckily, we find

$$A_4 = \frac{9}{5} \int_0^1 dx \frac{K_c(x)K_c(1-x)}{\sqrt{1-x}}, \quad (69)$$

$$B_4 = \frac{3\sqrt{3}}{4} \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}}. \quad (70)$$

We stress the tremendous simplification obtained by going from the usual description with elliptic integrals (58)-(62) to the “cubic” version (68)-(70).

4. Four-loop results

For the sake of brevity we define the following constants

$$C = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = 7.396\,099\,534\,768\,919\,553\,449\,114\,417\,961\,526\,519\,642\dots, \quad (71)$$

$$\mathcal{D} = \int_0^1 dx \frac{K_c(x)K_c(1-x)}{\sqrt{1-x}} = 4.860\,756\,383\,778\,595\,063\,430\,474\,772\,965\,586\,029\,529\dots, \quad (72)$$

$$E = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = 2.376\,887\,326\,184\,666\,003\,152\,855\,958\,761\,330\,926\,023\dots. \quad (73)$$

Now we look for relations between the numerical values of the above constants C , \mathcal{D} and E and the numerical values of $S_4(D=2)$ Eq.(28) and $S_4(D=4-2\epsilon)$ Eq.(29). We find that

$$S_4(-1, 1, 1, 1, 1, 1, D=2) = \pi\sqrt{3}C, \quad (74)$$

or, alternatively,

$$S_4(-1, 1, 1, 1, 1, 1, D=2) = \frac{4}{3}\pi B_4; \quad (75)$$

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we note the appearance of the factor $4\pi/3$, similar to the appearance of $4\pi/\sqrt{15}$ in Eq.(7). By using the integer-relation search program PSLQ¹⁸ we have been able to fit the numerical result of Eq.(29) with the analytical expression

$$S_4(-1, 1, 1, 1, 1, 1, D = 4 - 2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + c_0 + O(\epsilon) ,$$

$$c_0 = \frac{\pi\sqrt{3}}{240} (297C - 1477E) - \frac{2320981}{20736} . \quad (76)$$

The equalities Eq.(74) and Eq.(76) are the main result of this paper; they have been checked up to 2400 digits of precision. Note that constant \mathcal{D} does not appear in Eq.(74) and Eq.(76).

5. Four-dimensional lattice integrals

Considering lattice perturbation theory, at one loop level one finds these integrals¹⁵

$$Z_0 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{4 \sum_{\lambda=1}^4 \sin^2(k_{\lambda}/2)} = 0.154\,933\,390\,231\,060\,214\dots , \quad (77)$$

and

$$Z_1 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin^2(k_1/2) \sin^2(k_2/2)}{\sum_{\lambda=1}^4 \sin^2(k_{\lambda}/2)} = 0.107\,781\,313\,539\,874\,001\dots , \quad (78)$$

Some years ago, while visiting the Department of Physics of Parma, York Schröder pointed out to the author that, whether 3-loop g -2 was known in analytical form, no analytical result was known for the “simple” lattice 1-loop tadpole Z_0 . Puzzled by this fact, and noting that Z_0 can be reduced to a triple elliptic integral, we have tried to relate the numerical values of Z_0 and Z_1 to the new constants C , \mathcal{D} and E . Working with only 10-digits precision numbers we have discovered numerically that

$$\frac{S_4(-1, 1, 1, 1, 1, 1, D = 2)}{\pi^4 Z_0} \approx 8/3 . \quad (79)$$

That is

$$Z_0 \pi^3 = \frac{3\sqrt{3}}{8} C . \quad (80)$$

By using again PSLQ, we have also found that

$$Z_1 \pi^3 = -\frac{\sqrt{3}}{20} (3C + 7E) + \frac{\pi^3}{4} - \frac{\pi}{3} . \quad (81)$$

Values of Z_0 and Z_1 with 400 digits of precision are quoted in Ref. 15. Very kindly, York Schröder provided us 16000-digits values. By using these numbers, we have checked Eq.(80) and Eq.(81) up to 2400 digits of precision, the maximum precision of our values of C and E .

We note also that the integral (77) can be rewritten into the so-called Watson integral in 4-dimensions (see Refs. 16, 17)

$$u(4) = \frac{4}{(2\pi)^4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dk_1 dk_2 dk_3 dk_4}{4 - \cos k_1 - \cos k_2 - \cos k_3 - \cos k_4} = 8 Z_0 . \quad (82)$$

From Eq.(80) one obtains

$$u(4)\pi^3 = 3\sqrt{3}C . \quad (83)$$

6. Conclusions

In this paper we have at last identified beyond any reasonable doubt the analytical constants which appear in the simplest non-trivial 4-loop g -2 master integral. We have also discovered that the *same* constants appear in some 4-dimensional lattice integrals. Clearly we still do not know a rigorous proof of these relations, but, once the form of the results is known, we hope that proofs will be easier to find (see the very recent papers^{6,7,8}).

6.1. Note on Ref. 6

While we were completing this paper, kindly David Broadhurst sent us a copy of his new paper⁶. In that paper, several elliptic integral evaluations of Bessel moments are performed. In particular, a proof of our Eq.(5) and Eq.(22) is given, as well as of Eq.(44) and Eq.(46). Constants analogous to our A and B are found, $c_{4,0} = 2\pi A$ and $s_{4,0} = B$, as well as $t_{6,1} = A_4/8$ and $s_{6,1} = S_4(D=2)/16$. In addition, several relations between Bessel moments are found, and some evaluations of double elliptic integrals are done.

7. Acknowledgements

We thanks Enrico Onofri and Giuseppe Amoretti for helping us to retrieve the unpublished 3-loop results Eqs.(17)-(19) from a remote off-line computer, on occasion of the talk¹¹.

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